JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **90**, 207–208 (1982)

The Largest and the Smallest Characteristic Roots of a Positive Definite Matrix

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DEDICATED TO A. N. OSTROWSKI ON THE OCCASION OF HIS 90TH BIRTHDAY

1. Introduction

The purpose of this paper is to obtain bounds on the largest and the smallest characteristic roots of a positive definite matrix using the trace. We employ some well-known inequalities.

In Section 2 we give the method and in Section 3 we point out how the same method may be used for the smallest characteristic root using the inverse matrix.

2. Positive Definite Matrices

Let A be a positive definite matrix. It follows that all the characteristic roots are real and positive. Furthermore, we know that the characteristic roots of the powers of A are the corresponding powers of the characteristic roots of A. We also know that the trace, the sum of the elements along the main diagonal, is the sum of the characteristic roots.

Consider the sequence $A, A^2, \ldots, A^n, \ldots$. Let us define

$$U_n = \mathrm{Tr}(A^n).$$

Then we have

$$(U_n + 1)/U_n \leqslant \lambda_1 \leqslant U_n^{1/n}.$$

Here λ_1 is the largest characteristic value.

A simple application of the Cauchy–Schwarz inequality shows that the lower bound is monotone increasing. A simple result from the theory of inequalities shows that the upper bound is monotone decreasing.

0022-247X/82/110207-02$02.00/0

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3. Discussion

The same method may be employed to obtain the bound for the smallest characteristic root using the inverse matrix.

Analogous results may be obtained for integral equations with symmetric kernels.